Type Indeterminacy: A Model of the KT(Kahneman–Tversky)-Man*

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Abstract

In this paper we propose to use the mathematical formalism of Quantum Mechanics to capture the idea that agents' preferences, in addition to being typically uncertain, can also be *indeterminate*. They are determined (i.e., realized, and not merely revealed) only when the action takes place. An agent is described by a *state* that is a superposition of potential types (or preferences or behaviors). This superposed state is projected (or "collapses") onto one of the possible behaviors at the time of the interaction. In addition to the main goal of modelling uncertainty of preferences that is not due to lack of information, this formalism seems to be adequate to describe widely observed phenomena of noncommutativity in patterns of behavior. Two applications of the model are proposed. The first addresses the phenomenon of cognitive dissonance, and the second, framing effects. Experimental tests of the theory are suggested.

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1 Introduction

It has recently been proposed that models of quantum games can be used to study how the extension of classical moves to quantum ones (i.e., complex linear combinations of classical moves) can affect the analysis of a game. For example Eisert et al. (1999) show that allowing the players to use quantum strategies in the Prisoners' Dilemma is a way of escaping the well-known 'bad feature' of this game.[1] From a game-theoretical point of view the approach consists in changing the strategy spaces, and thus the interest of the results lies in the appeal of these changes.[2]

This paper also proposes to use the mathematical formalism of Quantum Mechanics but with a different intention: to model uncertain preferences. The basic idea is that the Hilbert space model of Quantum Mechanics can be thought of as a very general contextual predictive tool particularly well suited to describing experiments in psychology or in "revealing" preferences.

The well-established Bayesian approach suggested by Harsanyi to model incomplete information consists of a chance move that selects the types of the players and informs each player of his own type. For the purposes of this paper, we underline the following essential implication of this approach: all uncertainty about a player's type exclusively reflects the others player's incomplete knowledge of it. This follows from the fact that a Harsanyi type is fully determined. It is a complete well-defined characteristic of a player that is known to him. Consequently, from the point of view of the other players, uncertainty as to the type can only be due to lack of *information.* Each player has a probability distribution over the type of the other players, but his own type is fully determined and is known to him.

This brings us to the first important point at which we depart from the classical approach: we propose that in addition to informational reasons, the uncertainty as to preferences is due to *indeterminacy*: prior to the moment a player acts, his (behavior) type is indeterminate. The *state* representing the player, is a *superposition* of potential types. It is only at the moment

[1] In the classical version of the dilemma, the dominant strategy for both players is to defect and thereby to do worse than if they had both decided to cooperate. In the quantum version, there are a couple of quantum strategies that are both a Nash equilibrium and Pareto optimal and whose payoff is one of joint cooperation.

[2] This approach is closely related to quantum computing. It relies on the use of a sophisticated apparatus to exploit q-bits' property of entanglement in mixed strategies.

when the player selects an action that a specific type is actualized.[3] It is not merely revealed but rather determined in the sense that prior to the choice, there is an irreducible multiplicity of potential types. Thus we suggest that in modelling a decision situation, we do not assume that the preference characteristics can always be fully known with certainty (neither to the decision-maker nor even to the analyst). Instead, what can be known is the state of the agent: a vector in a Hilbert space which encapsulates all existing information to predict how the agent is expected to behave in different decision situations.

This idea, daringly imported from Quantum Mechanics to the context of decision and game theory, is very much in line with Tversky and Simonson (Kahneman and Tversky 2000) according to whom "There is a growing body of evidence that supports an alternative conception according to which preferences are often constructed – not merely revealed – in the elicitation process. These constructions are contingent on the framing of the problem, the method of elicitation, and the context of the choice". This view is also consistent with that of cognitive psychology, which teaches one to distinguish between objective reality and the proximal stimulus to which the observer is exposed, and to further distinguish between those and the mental representation of the situation that the observer eventually constructs. More generally, this view fits in with the observation that players (even highly rational ones) may act differently in game theoretically equivalent situations that differ only in seemingly irrelevant aspects (framing, prior unrelated events, etc.). Our theory as to why agents act differently in game theoretically equivalent situations is that they are not in the same state; i.e., they are not the same agents: (revealed) preferences are contextual because of (intrinsic) indeterminacy.

The basic analogy with Physics, which makes it appealing to adopt the mathematical formalism of Quantum Mechanics to the social sciences, is the following: we view an observed play, decisions, and choices as something similar to the result of a *measurement* (of the player's type). A decision situation is then similar to an experimental setup to measure the player's type. It is modeled as an operator (called *observable*), and the resulting behavior as an eigenvalue of that operator. The analogy to the non-commutativity of observables (a very central feature of Quantum Mechanics) is, in many empirical phenomena, like the following well-known experiment conducted by

[3] The associated concept of irreducible uncertainty which is the essence of indeterminacy, is formally defined in Section 2 of the paper.

Leon Festinger (the father of the theory of cognitive dissonance see, e.g., Festinger 1957). In that experiment people were asked to perform a very boring task. They would sort a batch of spools into lots of twelve and turn a square pegs a quarter turn to the left. They were then told that one subject was missing and asked to convince a potential female subject in the waiting room to participate. In one group they were offered \$1, and in the other group \$20, for expressing enthusiasm for the task. Some refused, but others accepted. Those who accepted for \$20, later admitted that they thought the task was dull. Those who accepted for \$1 maintained that the task was enjoyable. The experiment aimed at showing that attitudes change as a response to cognitive dissonance. The dissonance faced by those who were paid \$1 was between the cognition of being a 'good guy' and of being ready to lie for a dollar. Changing one's attitude to the task resolves the dissonance.[4] Similar phenomena have been documented in hazardous industries, with employees showing very little caution in the face of danger. Here too, experimental and empirical studies (e.g., Daniel Ben–Horing 1979) exhibited attitude changes among employees following their decision to work in a hazardous industry. More generally, suppose that an agent is subject to the same decision situation in two different contexts (the contexts may vary with respect to the decision situations that precede the investigated one, or with respect to the framing in the presentation of the decision, (cf. Selten 1998)). If we do not observe the same decision in the two contexts, then the classical approach is to say that the two decision situations *are not the same*; they should be modeled so as to incorporate the context. In many cases, however, such an assumption, i.e., that the situations are different, is difficult to justify. And so, the standard theory leaves a host of behavioral phenomena unexplained: the behavioral anomalies (cf. McFadden 1999).

In contrast, we propose that the observed decisions *are not taken by an agent in the same state*. The context, e.g., a past decision situation, is viewed as an operator that does not commute with the operator associated with the investigated decision situation; its operation on the agent has changed his state. As in Quantum Mechanics, the phenomenon of non-commutativity of decision situations (measurements) leads us to conjecture that an agent's preferences are represented by a state that is indeterminate and gets determined (with respect to any particular type characteristics) in the course of

[4]Those offered \$20 did not face such a tension because \$20 (in the fifties) was viewed as a sufficient motivation for a little lie.

interaction with the environment. Our approach allows us to go beyond the cognitive dissonance (CD) argument. According to the CD theory *cognitive coherence* is a basic human need (like food). Accordingly, CD has a 'drive-like' property (like hunger) that induces non-commutativity in behaviors and attitudes. However, the theory cannot explain why dissonance arises in the first place. As an illustration of the general approach we will show in Section 3 that a Hilbert space model of preferences can provide an appropriate framework in which to present and study CD–like phenomena.

The objective of this paper is to propose a theoretical framework that extends the Bayes–Harsanyi model to accommodate various forms of the so-called behavioral anomalies. We attempt to provide a model for the KT(Kahneman–Tversky)–man as opposed to what McFadden calls the 'Chicago man' (McFadden 1999). Our work is related to Random Utility Models (RUM) as well as to Behavioral economics. RUM models have proven very useful tools for explaining and predicting deviations from standard utility models. However, the RUM cannot accommodate the kind of drastic and systematic deviations characteristic of the KT–man. These models are based on an hypothesis of 'the primacy of desirability over availability' and they assume stable taste templates. In RUM, preferences are non-contextual by construction, while in the proposed Hilbert Space Model (HSM), (actualized) preferences are contextual. Behavioral economics has contributed a wide variety of theories (see Camerer 1997). Often the proposed explanations address a very specific deviation (e.g., 'trade off contrast' or 'extremeness aversion' (Kahneman and Tversky 2000). Important insights have been obtained by systematically investigating the consequences on utility maximization of 'fairness concerns' (Rabin 1993), 'cost of self-control' (Gul and Pesendorfer 2001) or 'concerns for self-image' (Benabou and Tirole 2002). Yet, other explanations appeal to bounded rationality, e.g., 'superficial reasoning' or 'choice of beliefs' (Selten 1998, Akerlof and Dickens 1982). In contrast, the HSM is a framework model that addresses structural properties of preferences, i.e., their intrinsic indeterminacy.

In section 2, we present the framework and some basic notions of quantum theory. In Section 3, we develop applications of the theory to social sciences. Two experiments to test the theory are suggested. Concluding remarks are gathered in the last section. The appendix provides a brief exposition of some basic concepts of quantum mechanics.

2 The basic framework

In this section we present the basic notions of our framework. They are heavily inspired by the mathematical formalism of Quantum Mechanics (see e.g., Cohen-Tannundji, Dui, Laloë 1973 and Cohen 1989) from which we also borrow the notation.

2.1 The notions of state and superposition

The object of our investigation is individual choice behavior, which we interpret as the revelation of an agent's preferences in what we call a *Decision Situation* (DS). In this paper we focus on non-repeated non-strategic decision situations. Examples of such DSs include the choice between buying a Toshiba or a Compaq laptop, the choice between investing in a project or not, the choice between a sure gain of \$100 or a bet with probability 0.5 to win \$250 and 0.5 to win \$0, etc. When considering games, we view them as decision situations from the perspective of a single player.[5]

An agent is represented by a *state* which captures the agent's expected behavior in the decision situation under consideration. Mathematically, a state $|\psi\rangle$ is a vector in a Hilbert space $\mathcal{H}$ of finite or countably infinite dimensions, over the field of the real numbers $\mathbb{R}$.[6] The relationship between $\mathcal{H}$ and a decision situation will be specified later. For technical reasons related to the probabilistic content of the state, the vector that represents it has to be of length one, that is, $\langle\psi\,|\psi\rangle^2 = 1$ (where $\langle\cdot\,|\cdot\rangle$ denotes the inner product in $\mathcal{H}$). So all vectors of the form $\lambda\,|\psi\rangle$ where $\lambda \in \mathbb{R}$, correspond to the same state, which we represent by a vector of length one.

A key ingredient in the formalism of indeterminacy is *the principle of superposition.* This principle states that the linear combination of any two states is itself a possible state.[7] Consider two states $|\varphi_1\rangle\,, |\varphi_2\rangle \in \mathcal{H}$, then

[5]All information (beliefs) and strategic considerations are embedded in the definition of the choices. Thus the agent's play of C is a play of C given, e.g., his information (knowledge) about the opponent.

[6]In Quantum Mechanics the number field is that of complex numbers. However, for our purposes the field of real numbers provides the structure and the properties needed (see e.g. Beltrametti and Cassinelli 1981 and Holland 1995). Everything we present in the appendix (Elements of quantum mechanics) remains true when we replace Hermitian operators with real symmetric operators.

[7]We use the term state to refer to 'pure state'. Some people use the term state to refer to mixture of pure states. A mixture of pure states combines indeterminacy with elements

$|\psi\rangle \in \mathcal{H}$ where $|\psi\rangle = \lambda_1 |\varphi_1\rangle + \lambda_2 |\varphi_2\rangle$ for any $\lambda_1, \lambda_2 \in \mathbb{R}$, with $\lambda_1^2 + \lambda_2^2 = 1$. The principle of superposition implies that, unlike the Harsanyi type space, the state space is non-Boolean.[8]

2.2 The notions of observable and of measurement

A decision situation is defined by the set of alternative choices available to the agent. When an agent selects an action out of a set corresponding to a decision situation, we say that he 'plays' the DS. To every Decision Situation A, we associate an *observable,* namely, a specific symmetric operator on $\mathcal{H}$ which, for notational simplicity, we also denote by A. If A is the only decision situation we consider, we can assume that its eigenvectors, which we denote by $|1_A\rangle, |2_A\rangle, ..., |n_A\rangle$, all correspond to different eigenvalues, denoted by $1_A, 2_A, ..., n_A$ respectively.

$$A\,|k_A\rangle = k_A\,|k_A\rangle\,,\quad k = 1, ..., n.$$

As A is symmetric, there is a unique orthonormal basis of the relevant Hilbert space $\mathcal{H}$ formed with its eigenvectors. The basis $\{|1_A\rangle, |2_A\rangle, ..., |n_A\rangle\}$ is the unique orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of A. It is thus possible to represent the agent's state as a superposition of the vectors of this basis:

$$|\psi\rangle \ = \sum_{k=1}^{n} \lambda_k\,|k_A\rangle\,, \tag{1}$$

where $\lambda_k \in \mathbb{R}, \forall k \in \{1, ..., n\}$ and $\sum\limits_{k=1}^{n} \lambda_k^2 = 1$.

The expansion (1) can also be written as

$$\mathcal{H} = \mathcal{H}_{1_A} \oplus, ..., \oplus \mathcal{H}_{n_A},\ \mathcal{H}_{i_A} \perp \mathcal{H}_{j_A},\ i \neq j, \tag{2}$$

of incomplete information. They are represented by so called density operators.

[8]The distributivity condition defining a Boolean space is dropped for a weaker condition called ortho-modularity. The basic structure of the state space is that of a logic, i.e., an orthomodular lattice. For a good presentation of Quantum Logic, a concept introduced by Birkhoff and Von Neuman (1936), and further developed by Mackey (2004, 1963), see Cohen (1989).

where $\oplus$ denotes the direct sum of the subspaces $\mathcal{H}_{1_A}, ..., \mathcal{H}_{n_A}$ spanned by $|1_A\rangle, ..., |n_A\rangle$ respectively.[9] Or, equivalently, we can write $I_{\mathcal{H}} = P_{1_A} +, ..., +P_{n_A}$ where P_{i_A} is the projection operator on $\mathcal{H}_{i_A}$ and $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$.

A decision situation A can be thought of as an experimental setup where the agent is invited to choose a particular action among all possible actions in that decision situation. The actual implementation of the experiment is represented by a *measurement* of the associated observable A. According to the so-called Reduction Principle (see Appendix), the result of such a measurement can only be one of the n eigenvalues of A. If the result is m_A, i.e., the player selects action m_A, the superposition $\sum \lambda_i |i_A\rangle$ "collapses" onto the eigenvector associated with the eigenvalue m_A. The initial state $|\psi\rangle$ is projected into the subspace $\mathcal{H}_{m_A}$(of eigenvectors of A with eigenvalue m_A). The probability that the measurement yields the result m_A is equal to $\langle m_A |\psi\rangle^2 = \lambda_m^2$,[10] i.e., the square of the corresponding coefficient in the superposition. The coefficients themselves, called 'amplitudes of probability' play a key role in studying sequences of measurements (see Section 2.3). As usual, we will interpret the probability of m_A either as the probability that one agent in state $|\psi\rangle$ selects action m_A or as the proportion of the agents who will make the choice m_A in a population of many agents, all in the state $|\psi\rangle$.

In our theory an agent is represented by a state. We shall also use the term type to denote a state degenerated to one eigenvector, say $|m_A\rangle$. An agent in this state is said to be of type m_A. An agent in a general state $|\psi\rangle$ is hence a superposition of all types relevant to the DS under consideration. Our notion of type is closely related to the notion used by Harsanyi. A type captures all the agent's characteristics (taste, subjective beliefs) of relevance for *uniquely* predicting the agent's choice *in a given situation*. In contrast to Harsanyi we shall not assume that there exists an exhaustive description of the agent that enables us to uniquely determine the agent's choice in all possible decision situations *simultaneously*. Instead, our types are characterized by an irreducible uncertainty that is revealed when the agent is confronted with a

[9] That is, for $i \neq j$ any vector in $\mathcal{H}_{i_A}$ is orthogonal to any vector in $\mathcal{H}_{j_A}$ and any vector in $\mathcal{H}$ is a sum of n vectors, one in each component space.

[10] For simplicity we assume that all eigenvalues are 'non-degenerated'. We return to this issue below.

sequence of choices (see Section 2.3.2 below for a formal characterization of irreducible uncertainty).

Remark: Clearly, when only one DS is considered, the above description is equivalent to the traditional probabilistic representation of an agent by a probability vector $(\alpha_{1,...,}\alpha_n)$ in which α_k is the probability that the agent will choose action k_A and $\alpha_k = \lambda_k^2$ for $k = 1, ..., n$. The advantage of the proposed formalism consists in enabling us to study several decision situations and the interaction between them.

2.3 More than one Decision Situation

When studying more than one DS, say A and B, it turns out that a key question is whether the corresponding observables are commuting operators in $\mathcal{H}$, i.e., whether $AB = BA$. The question of whether two DSs can be represented by two commuting operators is an empirical one. Before commenting on this feature we will first study its mathematical implications.

2.3.1 Commuting Decision Situations

Let A and B be two DSs. If the corresponding observables commute then there is an orthonormal basis of the relevant Hilbert space $\mathcal{H}$ formed by eigenvectors common to both A and B. Denote by $|i\rangle$ (for $i = 1, ..., n$) these basis vectors. We have

$$A\,|i\rangle = i_A\,|i\rangle \text{ and } B\,|i\rangle = i_B\,|i\rangle\,.$$

In general, the eigenvalues can be degenerated (i.e., for some i and j, $i_A = j_A$ or $i_B = j_B$). Any normalized vector $|\psi\rangle$ of $\mathcal{H}$ can be written in this basis:

$$|\psi\rangle = \sum_i \lambda_i\,|i\rangle\,,$$

where $\lambda_i \in \mathbb{R}$, and $\sum_i \lambda_i^2 = 1$. If we measure A first, we observe eigenvalue i_A with probability

$$p_A\,(i_A) = \sum_{j;j_A=i_A} \lambda_j^2. \tag{3}$$

If we measure B first, we observe eigenvalue j_B with probability $p_B(j_B) = \sum_{k;k_B=j_B} \lambda_k^2$. After B is measured and the result j_B obtained, the state $|\psi\rangle$ is projected into the eigensubspace $\mathcal{E}_{j_B}$ spanned by the eigenvectors of B associated with j_B. More specifically, it collapses onto the state:

$$\left|\psi_{j_B}\right\rangle = \frac{1}{\sqrt{\sum_{k;k_B=j_B} \lambda_k^2}} \sum_{k;k_B=j_B} \lambda_k |k\rangle$$

(the factor $\frac{1}{\sqrt{\sum_{k;k_B=j_B} \lambda_k^2}}$ is necessary to make $\left|\psi_{j_B}\right\rangle$ a unit vector).

When we measure A on the agent in the state $\left|\psi_{j_B}\right\rangle$, we obtain i_A with probability

$$p_A(i_A|j_B) = \frac{1}{\sum_{k;k_B=j_B} \lambda_k^2} \sum_{\substack{k;k_B=j_B \\ \text{and } k_A=i_A}} \lambda_k^2.$$

So when we measure first B and then A, the probability of observing the eigenvalue i_A is $p_{AB}(i_A) = \sum_j p_B(j_B)\, p_A(i_A|j_B)$:

$$\begin{aligned} p_{AB}(i_A) &= \sum_j \frac{1}{\sum_{k;k_B=j_B} |\lambda_k|^2} \sum_{k;k_B=j_B} \lambda_k^2 \sum_{\substack{l;l_B=j_B \\ \text{and } l_A=i_A}} \lambda_l^2 \\ &= \sum_j \sum_{\substack{l;l_B=j_B \\ \text{and } l_A=i_A}} \lambda_l^2 = \sum_{l;l_A=j_A} \lambda_l^2. \end{aligned}$$

Hence, $p_{AB}(i_A) = p_A(i_A)\,, \forall i$, and similarly $p_{BA}(j_B) = p_B(j_B)\,, \forall j$.

When dealing with commuting observables it is meaningful to speak of measuring them simultaneously. Whether we measure first A and then B or first B and then A, the probability distribution on the joint outcome is $p(i_A \wedge j_B) = \sum_{\substack{k;k_B=j_B \\ \text{and } k_A=i_A}} \lambda_k^2$, so $(i_A\,,\, j_B)$ is a well-defined event. Formally, this implies that the two DSs can be merged into a single DS with a vector-valued measurement outcome, i.e., a value in A and a value in B. To each eigenvalue of the merged observable we associate a type that captures all the characteristics of the agent relevant to his choices (one in each DS).

Remark: Note that again, as in the case of a single DS, for two such DSs our model is equivalent to a standard (discrete) probability space in which the elementary events are $\{(i_A, j_B)\}$ and $p(i_A \wedge j_B) = \sum_{\substack{k;k_B=j_B \\ \text{and } k_A=i_A}} \lambda_k^2$. In particular, in accordance with the calculus of probability we see that the conditional probability formula holds:

$$p_{AB}(i_A \wedge j_B) = p_A(i_A)\ p_B(j_B|i_A).$$

For example, consider the following two decision problems. Let A be the decision situation of choosing between a week vacation in Tunisia and a week vacation in Italy. And let B be the choice between buying 1000 euros of shares in Bouygues Telecom or in Deutsche Telecom. It is quite plausible that A and B commute, but whether or not this is in fact the case is of course an empirical question. If A and B commute we expect a decision on portfolio (B) not to affect decision-making regarding the location for vacation (A). And thus the order in which the decisions are made does not matter.

Note that the commutativity of the observables does not exclude statistical correlations between observations. To see this, consider the following example in which A and B each have two degenerated eigenvalues in a four dimensional Hilbert space. Denote by $|i_A j_B\rangle$ $(i = 1, 2,\ j = 1, 2)$ the eigenvector associated with eigenvalues i_A of A and j_B of B, and let the state $|\psi\rangle$ be given by

$$|\psi\rangle = \sqrt{\tfrac{3}{8}}\,|1_A 1_B\rangle + \sqrt{\tfrac{1}{8}}\,|1_A 2_B\rangle + \sqrt{\tfrac{1}{8}}\,|2_A 1_B\rangle + \sqrt{\tfrac{3}{8}}\,|2_A 2_B\rangle$$

Then, $p_A\left(i_A|1_B\right) = \frac{\frac{3}{8}}{\frac{3}{8}+\frac{1}{8}} = \frac{3}{4}$ (for $i = 1$ or $i = 2$).

So if we first measure B and find, say, 1_B, it is much more likely (with probability $\frac{3}{4}$) that when measuring A we will find 1_A rather than 2_A (with probability $\frac{1}{4}$). In other words, the statistical correlation is informational: information about one of the observables is relevant to the prediction of the value of the other. But the two *interactions* (measurements) do not affect each other, i.e., the distribution of the outcomes of the measurement of A is the same whether or not we measure B first.

2.3.2 Non-commuting Decision Situations

It is when we consider decision situations associated with observables that do not commute that the predictions of the HSM differ from those of the probabilistic model. In such a context, the quantum probability calculus $\left(p\left(\langle i_A\,|\psi\rangle\right) = \langle i_A\,|\psi\rangle^2\right)$ generates cross-terms also called interference term. These cross-terms are the signature of indeterminacy. In the next section we demonstrate how this feature of the HSM captures the phenomenon of cognitive dissonance as well as that of framing.

For simplicity, assume that the two decision situations A and B have the same number n of possible choices, which means that the observables A and B have (non-degenerated) eigenvalues $1_A, 2_A, ..., n_A$ and $1_B, 2_B, ..., n_B$ respectively and each one of the sets of eigenvectors $\{|1_A\rangle, |2_A\rangle, ..., |n_A\rangle\}$ and $\{|1_B\rangle, |2_B\rangle, ..., |n_B\rangle\}$ is an orthonormal basis of the relevant Hilbert space. Let $|\psi\rangle$ be the initial state of the agent

$$|\psi\rangle \; = \sum_{i=1}^{n} \lambda_i |i_A\rangle = \sum_{j=1}^{n} \nu_j \, |j_B\rangle \, .$$

We note that since each set of eigenvectors of the respective observables forms a basis of the state space there exists a unitary operator S such that

$$|\psi\rangle \; = (\lambda_1, ..., \lambda_N) \, S \begin{pmatrix} |1_B\rangle \\ . \\ . \\ |n_B\rangle \end{pmatrix} .$$

S is a basis transformation $n \times n$ matrix with elements $\langle j_B \, | i_A \rangle$. In the next section we show that this matrix plays an important role in practical applications of the theory.[11] For the ease of presentation we write $\mu_{ij} = \langle j_B \, | i_A \rangle$ and we can write $|j_B\rangle$ as

$$|j_B\rangle = \sum_{i=1}^{n} \mu_{ij} |i_A\rangle,$$

implying

$$|\psi\rangle \; = \sum_{j=1}^{n} \sum_{i=1}^{n} \nu_j \mu_{ij} |i_A\rangle.$$

If the agent plays A directly, he chooses i_A with probability $p_A(i_A) = \left(\sum\limits_{j=1}^{n} \nu_j \mu_{ij} \right)^2$.

If he first plays B, he selects action j_B with probability ν_j^2 and his state is projected onto $|j_B\rangle$. The agent then selects action i_A in decision situation A with probability μ_{ij}^2. So the (ex-ante) probability for i_A is $p_{AB}(i_A) = \sum\limits_{j=1}^{n} \nu_j^2 \mu_{ij}^2$,

[11]Conversely, we have that S^{-1} transforms the basis formed by the eigenvectors of the B observable into the basis formed by the eigenvectors of A.

which is in general different from $\left(\sum_{j=1}^{n}\nu_j\mu_{ij}\right)^2$. Playing B first changes the way A is played. The difference stems from the so-called *interference terms*

$$
\begin{aligned}
p_A(i_A) &= \left(\sum_{j=1}^{n}\nu_j\mu_{ij}\right)^2 = \sum_{j=1}^{n}\nu_j^2\mu_{ij}^2 + \underbrace{2\sum_{j\neq j'}\left[\left(\nu_{j'}\mu_{ij}\right)\left(\nu_j\mu_{ij'}\right)\right]}_{\text{Interference term}} \\
&= p_{AB}(i_A) + \text{interference term}
\end{aligned}
$$

The interference term is the sum of cross-terms involving the amplitudes of probability (the Appendix provides a description of interference effects in Physics).

Some intuition about interference effects may be provided using the concept of 'propensity' due to Popper (1992). Imagine an agent's mind as a system of propensities to act (corresponding to different possible actions). As long as the agent is not required to choose an action in a given DS, the corresponding propensities coexist in his mind; the agent has not "made up his mind". A decision situation operates on this state of 'hesitation' to trigger the emergence of a single type (of behavior). But as long as alternative propensities are present in the agent's mind, they affect choice behavior by increasing or decreasing the probability of the choices under investigation.

An illustration of this kind of situation may be supplied by the experiment reported in Knetz and Camerer (2000). The two DSs studied are the Weak Link (WL) game and the Prisoners' Dilemma (PD) game.[12] They compare the distribution of choices in the Prisoners' Dilemma (PD) game when it is preceded by a Weak Link (WL) game and when only the PD game is being played. Their results show that playing the WL game affects the play of individuals in the PD game. The authors appeal to an informational argument, which they call the "precedent effect".[13] However, they cannot explain the high rate of cooperation (37.5 %) in the last round of the PD

[12]The Weak Link game is a type of coordination game where each player picks an action from a set of integers. The payoffs are defined in such a manner that each player wants to select the minimum of the other players but everyone wants that minimum to be as high as possible.

[13]The precedent effect hypothesis is as follows: "The shared experience of playing the efficient equilibrium in the WL game creates a precedent of efficient play strong enough to

game (Table 5, p. 206). Instead, we propose that the WL and the PD are two DS that do not commute. In such a case we expect a difference in the distributions of choices in the (last round of the) PD depending on whether or not it was preceded by a play of the WL or another PD game.

Remark: In this case, A and B cannot have simultaneously defined values: the state of the agent is characterized by *irreducible uncertainty*. Equivalently, and in contrast with the commuting case, two non-commuting observables cannot be merged into one observable. There is no probability distribution on the event 'to have the value i_A for A and the value j_B for B.' The conditional probability formula does not hold:

$$p_A(i_A) \neq \sum_{j=1}^{n} p_B(j_B)\ p(i_A|j_B).$$

Indeed, $\left(\sum_{j=1}^{n}\nu_j\mu_{ij}\right)^2 = p_A(i_A) \neq \sum_{j=1}^{n} p_B(j_B)\ p(i_A|j_B) = \sum_{j=1}^{n}\nu_j^2\mu_{ij}^2.$

As G. W. Mackey expresses it "When A and B do not commute there are limitations to the degree to which the probability distribution of the corresponding observables may be simultaneously concentrated near to single points" (Mackey 2000, p. 78). In our context these limitations can be interpreted as reflecting the psychological coherence structure. When the agent knows what he prefers in one situation, he is 'doomed' to be hesitating in some other (non-commuting) decision situation.

3 HSM in the Social Sciences

In this section we apply the general framework to explain two instances of behavioral anomalies that have been extensively studied in the literature but under very different approaches.

3.1 The Act of Choice

A central feature of the HSM approach is that choices alter the state of the agent, which may imply non-commutativity of choice behavior. Of course,

(...) lead to cooperation in a finitely repeated PD game", Knetz and Camerer (2000 see p.206).

not all instances of non-commutativity in decision theory call for Hilbert space modelling. An agent's preferences between a long opera or a 'stay at home evening' are not expected to be the same *before* an afternoon party where she is invited to choose between whisky or champagne and *after* she has been drinking a few glasses. Theories of addiction also feature effects of past choices on future preferences. And in standard consumer theory, choices do have implications for future behavior, i.e., when goods are substitutes or complements. The above examples hardly qualify as behavioral anomalies, however. In the above mentioned cases we *do* expect future preferences to be affected by the choices. The Hilbert space model of preferences is useful when we expect choice behavior to be consistent with the standard probabilistic model, because nothing justifies a modification of preferences. Yet, actual behavior contradicts those expectations.

In the article "Maximization and the Act of Choice" (1997) Nobel laureate Amartya Sen argues for the need to account for the 'act of choice' as distinguished from the 'result of choice' in social and economic analysis. The phenomenon known as 'cognitive dissonance' which refers to a change in 'attitude' following a decision, is the most straightforward expression of the effect of the act of choice on future behavior. We propose below a HSM presentation of cognitive dissonance that features a dynamic effect of the act of choice in line with psychologists' 'forces of cognitive coherence'.

3.1.1 Dynamic decision-making

In this section we consider a dynamic decision problem involving a sequence of two decisions, each with two options: $A : \{a_1, a_2\}$ and $B : \{b_1, b_2\}$. We shall assume that the agent is forward-looking, i.e., he takes into account the effect of his current decision on his future preferences and decisions. This approach facilitates the comparison with Akerlof and Dickens (1982) who propose a cognitive dissonance model in which highly sophisticated agents choose beliefs to fit their preferences.[14]

The sequence of decisions is first A then B. As usual, we analyze the dynamic decision problem by backward induction. Any measurement of B

[14]Akerlof and Dickens (1981) allow workers to freely choose beliefs (about risk) so as to optimize utility which includes psychological comfort. They are fully aware of the way their subjective perception of the world is biased and yet they keep to the wrong views. They are nevertheless highly rational in the sense that when selecting beliefs, they internalize their effect their own subsequent bounded rational behavior.

can give only one of two eigenvalues of the corresponding operator: either b_1 or b_2. This leads us to define the two types as follows. When the agent chooses b_1 and the state collapses onto $|b_1\rangle$ we say that the agent is of the b_1-type defined by $b_1 \succ b_2$, i.e., he prefers b_1 to b_2; and similarly with the b_2-type: $b_2 \succ b_1$.

We now consider the first decision situation. We have assumed that the agent knows that decision situation A is followed by B and that the two DSs are linked (non-commuting). We also assume that the agent knows the correlation pattern, i.e., the S matrix (see Section 2.3.2). The elements of the S matrix are the square roots of the statistical correlations between the types associated with the choices in the respective DSs.[15] We shall denote the choices in A (when followed by B) as follows: $a_1(B)$, $a_2(B)$. The correlation matrix is

$$S = \begin{pmatrix} \langle b_1 | a_1(B)\rangle & \langle b_2 | a_1(B)\rangle \\ \langle b_1 | a_2(B)\rangle & \langle b_2 | a_2(B)\rangle \end{pmatrix}.$$

As in the general setting, we write the eigenvectors of A in terms of the eigenvectors of B (recall that in the quantum formalism $|\psi\rangle$ and $2|\psi\rangle$ represent the same state):

$$|a_1(B)\rangle = \langle b_1 | a_1(B)\rangle |b_1\rangle + \langle b_2 | a_1(B)\rangle |b_2\rangle \tag{4}$$

$$|a_2(B)\rangle = \langle b_1 | a_2(B)\rangle |b_1\rangle + \langle b_2 | a_2(B)\rangle |b_2\rangle . \tag{5}$$

By assumption the agent is forward-looking, so the choice of a_1 actualizes preferences with the following interpretation: a_1 and the 'lottery' in B associated with a_1(implied by (4)), is preferred to a_2 and its associated lottery (implied by (5)).[16] At the beginning of the choice sequence the agent

[15] This is clearly seen in eq. (4) below. The probability that type $a_1(B)$ will choose b_1 is given by the square of the coefficient of superposition, i.e., $\langle b_1 | a_1(B)\rangle^2$.

[16] The $a_1(B)-$type can be described in terms of preferences over 'bundles' where the second term is the associated lottery:

$$(a_1, (\alpha b_1 + (1-\alpha) b_2)) \succ (a_2, (\beta b_1 + (1-\beta) b_2))$$

where $\alpha = \langle b_1 | a_1(B)\rangle^2$ and $\beta = \langle b_1 | a_2(B)\rangle^2$. We use the term 'lottery' to capture the agent's indeterminacy in B. This is consistent with our results in Section 2.2, which show the equivalence of the HSM with the probabilistic model in the single DS case. At the time when the agent faces B, B is the only relevant DS.

is in a state of superposition:

$$|\psi\rangle = \lambda_1 |a_1(B)\rangle + \lambda_2 |a_2(B)\rangle, \; \lambda_1^2 + \lambda_2^2 = 1.$$

3.1.2 Cognitive dissonance

We now show how the above model captures cognitive dissonance (CD). The typical 'story' is that workers who choose employment in risky jobs behave as if they undervalued risk compared with workers who are not confronted with such a choice.

We define the options as follows. Let A be a decision about jobs; a_1 : take a job with a hazardous task (adventurous type), a_2 : stay in a safe task (habit-prone type). Let B be a decision about behavior at the risky workplace; the choices are b_1: use safety equipment (risk-averse type), b_2 : don't use safety equipment (risk-loving type).

First scenario: The hazardous task is introduced in an existing context. It is *imposed* on the workers. They are given *only* the choice to use or not to use safety equipment (B). We write the initial state of the worker as

$$|\psi\rangle = \lambda_1 |a_1(B)\rangle + \lambda_2 |a_2(B)\rangle, \; \lambda_1^2 + \lambda_2^2 = 1.$$

Substituting for (4) and (5) we write the state in the basis of the B operator as

$$\begin{aligned} |\psi\rangle &= [\lambda_1 \langle b_1| a_1(B)\rangle + \lambda_2 \langle b_1| a_2(B)\rangle] |b_1\rangle + \\ &\quad + [\lambda_1 \langle b_2| a_1(B)\rangle + \lambda_2 \langle b_2| a_2(B)\rangle] |b_2\rangle . \end{aligned}$$

The probability that a worker chooses to use safety equipment is

$$\begin{aligned} p_B(b_1) &= \langle b_1| \psi\rangle^2 = [\lambda_1 \langle b_1| a_1(B)\rangle + \lambda_2 \langle b_1| a_2(B)\rangle]^2 \\ &= \lambda_1^2 \langle b_1| a_1(B)\rangle^2 + \lambda_2^2 \langle b_1| a_2(B)\rangle^2 + 2\lambda_1\lambda_2 \langle b_1| a_2(B)\rangle \langle b_1| a_1(B)\rangle . \end{aligned}$$

Second scenario: First A then B. The workers choose between taking a new job with a hazardous task or staying with the current safe routine. Those who chose the new job then face the choice between adopting safety measures or not. Those who turn down the new job offer are asked to answer a questionnaire about their choice in the hypothetical case where they are confronted with a risky task. The ex-ante probability for observing b_1 is

$$\begin{aligned} p_{BA}(b_1) &= p_A(a_1(B))\,p_B(b_1|\,a_1(B)) + p_A(a_2(B))\,p_B(b_1|\,a_2(B)) \\ &= \lambda_1^2 \langle b_1|\,a_1(B)\rangle^2 + \lambda_2^2 \langle b_1|\,a_2(B)\rangle^2 . \end{aligned}$$

The empirically documented phenomenon of 'cognitive dissonance' can now be formulated as

$$p_{BA}(b_1) < p_B(b_1),$$

which occurs in our model when $2\lambda_1\lambda_2 \langle b_1|\,a_2(B)\rangle \langle b_1|\,a_1(B)\rangle > 0$. Before moving to a numerical example showing that interference effects may be quantitatively significant, we note that $p_{BA}(b_1)$ includes the probability of a choice of safety measures *both* in the group that chose the risky job and in the group that chose the safe job. This guarantees that we properly distinguish between the CD effect (change in attitude) and the selection bias.

Numerical example

Assume for simplicity that $|\psi\rangle = |b_1\rangle$ so everybody in the first scenario is willing to use the proposed safety equipment. Let $prob\,(a_1\,|\psi\,) = 0.25$, and $prob\,(a_2\,|\psi\,) = .75$ so $\langle b_1\,|a_1\rangle = \pm\sqrt{0.25}$, and $\langle b_1\,|a_2\rangle = \pm\sqrt{0.75}$. We now compute $prob_B(b_1)$ using the expanded form

$$\begin{aligned} \langle b_1\,|\psi\rangle^2 &= \left(\pm\sqrt{0.75}\,\langle b_1\,|a_1\rangle + \left(\pm\sqrt{0.25}\,\langle b_1\,|a_2\rangle\right)\right)^2 \\ &= 0.5625 + 0.0625 \pm 2\cdot 0.1875. \end{aligned}$$

Since $|\psi\rangle = |b_1\rangle$, we have $\langle b_1\,|\psi\rangle^2 = 1$ which implies that the interference effect is positive (the amplitudes $\langle b_1\,|a_1\rangle$ and $\langle b_1\,|a_2\rangle$ are of the same sign). We note that it amounts to a third of the probability.

Under the second scenario the probability for taking the safety measure is the sum of the conditional which is obtained immediately by substraction of the interference term

$$p_{BA}(b_1) = p_B(b_1) - \text{interference term} = 1 - 2\cdot 0.1875 = 0.625$$

The desired result obtains: $p_{BA}(b_1) < p_B(b_1)$.

Comments

The contribution of the indeterminacy approach is two-sided. First the HSM provides a model that explains the appearance of 'cognitive dissonance'.

Indeed, if coherence is such a basic need, as proposed by L. Festinger and his followers, why does dissonance occur in the first place? In the HSM 'dissonance' arises when resolving indeterminacy in the first DS because of the 'limitations' on possible psychological types (cf Mackey (2000) and Section 2.3.2). Second, the HSM features a *dynamic process* such that the propensity to use safety measures is actually altered (reduced) as a consequence of the *act of choice*. This dynamic effect of coherence (which arises when choosing in the second DS) is reminiscent of the psychologists' "drive-like property of dissonance" leading to a change in attitude. Coherence is expressed by the structure of the state space itself, i.e., by the correlation matrices that reflect psychological regularities.

3.2 Framing Effects

Providing a framework for thinking about framing effects is one of the important and promising applications of the indeterminacy approach in decision theory. Kahneman and Tversky (2002, p. xiv) define the framing effect using a two-steps (nonformal) model of the decision-making process. The first step corresponds to the construction of a representation of the decision situation. The second step corresponds to the evaluation of the choice alternatives. The crucial point is that *"the true objects of evaluation are neither objects in the real world nor verbal description of those objects; they are mental representations."* To capture this feature we suggest modelling of the process of constructing a representation in a way similar to the process of constructing preferences. This is consistent with the approach in psychology that treats attitudes, values (preferences), beliefs, and representations as mental objects of the same kind. We thus propose that the process of decision-making be modelled as a sequence of two observables (operators). A framing or FR is defined as a collection of alternative mental representations (of a decision situation). The corresponding observable is denoted $\widetilde{A}$, where the tilde is used to distinguish between operators associated with FRs and those associated with DSs. We model the process of constructing (selecting) a representation as the analogue of the 'measurement' of the operator corresponding to the framing proposed to the agent, i.e., the collapse onto one of the framing operators' eigenvectors (onto one of the alternative mental representations).[17]

[17] Interestingly, this approach is consistent with research in neurobiology that suggests that the perception of a situation involves operations in the brain very similar to those

In contrast with the types associated with a DS, the types associated with a FR (hereafter we call them FR-type) are not directly observable but they could, in principle, be elicited by a questionnaire.

The operators associated with FRs and DSs are defined within the same state space of representations/preferences. One interpretation is that representations and preferences have grown entangled through the activity of the mind i.e. in decision processes.[18] We now provide an illustration of this approach.

In Selten (1998) an experiment performed by Pruitt (1970) is discussed. Two groups of agents are invited to play a Prisoners' Dilemma. The game in the usual matrix form is presented to the first group but with the choices labelled 1 and 2 (instead of C and D, presumably to avoid associations with the suggestive terms 'cooperate' and 'defect'):

	$[C]$	$[D]$
$1_G(C)$	3 3	4 0
$2_G(D)$	0 4	1 1

The game is presented to the second group with payoffs decomposed as follows:

	For me	For him
1_G	0	3
2_G	1	0

The payoffs are the sums of what you take for yourself and what you get from the other player. Game theoretically it should make no difference whether the game is presented in matrix form or in decomposed form. However, an experiment by Pruitt shows that one observes dramatically more cooperation in the game when presented in decomposed form.

involved when chosing an action. 'To perceive is not only to combine, weigh, it is to select'. (Berthoz, 2003, p.10)

[18] In quantum physics the entanglement of the states of two spin-1/2 particles in the singlet spin state means that the two particles form a single system that cannot be factored out into two subsystems even when separated by a great distance (cf. the famous Einstein-Podolsky-Rosen (EPR) non-locality paradox).

Selten proposes a 'bounded rationality' explanation: players make a superficial analysis and do not perceive the identity of the game presented under the two forms. We propose that the agents after having been confronted with the matrix presentation are not in the same state as before and not in the same state as the agents that have been confronted with the decomposed form. The object of evaluation does not pre-exist the interaction between the agent and the framed decision situation. It is, as Kahneman and Tversky (2000) propose, constructed in the process of decision-making.

Using the framework developed in this paper, we call $\widetilde{A}$ the FR operator corresponding to the framing of the matrix presentation, $\widetilde{B}$ the one corresponding to the framing of the decomposed presentation, and G the DS operator associated with the game. G has two non-degenerated eigenvalues denoted 1_G and 2_G. The first step in the decision process corresponds to the selection of a mental representation of the decision situation. At this stage, the agent is subjected to a frame $\widetilde{A}$ or $\widetilde{B}$. Accordingly, before being presented with the DS in some form, the agent is described by a superposition of possible mental representations. For simplicity we assume that each frame only consists of two alternative representations. We can write $|\psi\rangle = \alpha_1 |\widetilde{a}_1\rangle + \alpha_2 |\widetilde{a}_2\rangle$ and $|\psi\rangle = \beta_1 \left|\widetilde{b}_1\right\rangle + \beta_2 \left|\widetilde{b}_2\right\rangle$. We propose the following description of the mental representations (this is meant only as a suggestive illustration)

$|\widetilde{a}_1\rangle$: G is perceived as an (artificial) small-stake game;

$|\widetilde{a}_2\rangle$: G is perceived by analogy as a real life PD-like situation (often occurring in a repeated setting).

$\left|\widetilde{b}_1\right\rangle$: G is perceived as a test of generosity;

$\left|\widetilde{b}_2\right\rangle$: G is perceived as a test of smartness.

Given any initial state, subjecting the agent to either frame leads to a collapse onto either one of the two mental representations associated with that frame. The state of the agent becomes some $|\widetilde{a}_i\rangle$ or $\left|\widetilde{b}_j\right\rangle$ $(i, j = 1, 2)$. In the basis of the decision situation $\{|1_G\rangle, |2_G\rangle\}$, the state of our agent is

$|\widetilde{a}_i\rangle = \gamma_{1i} |1_G\rangle + \gamma_{2i} |2_G\rangle$;

$\left|\widetilde{b}_j\right\rangle = \delta_{1j} |1_G\rangle + \delta_{2j} |2_G\rangle$.

We can now express the framing effect by the following difference:

$$p_{G\widetilde{A}}(i_G) \neq p_{G\widetilde{B}}(i_G), \quad i = 1, 2. \tag{6}$$

Using our result in Section 2.3.2 we get

$$p_{G\widetilde{A}}(1_G) = p_G(1_G) - 2\alpha_1\gamma_{11}\alpha_2\gamma_{12}$$
$$p_{G\widetilde{B}}(1_G) = p_G(1_G) - 2\beta_1\delta_{11}\beta_2\delta_{12},$$

where $p_G(1_G)$ is the probability of choosing 1 in an (hypothetical) unframed situation. So we have a framing effect whenever $2\alpha_1\gamma_{11}\alpha_2\gamma_{12} \neq 2\beta_1\delta_{11}\beta_2\delta_{12}$. The experimental results discussed in Selten (1998) obtain with $2\beta_1\delta_{11}\beta_2\delta_{12} < 2\alpha_1\gamma_{11}\alpha_2\gamma_{12}$. In general, unless the FR-types belonging to $\widetilde{A}$ are perfectly correlated with those belonging to $\widetilde{B}$ or the FR operators both commute with the decision operator, we have $p_{G\widetilde{A}}(1_A) \neq p_{G\widetilde{B}}(1_A)$.

Comments

Kahneman and Tversky suggest that prior to the choice, a representation of the decision situation must be constructed. The HSM provides a framework for 'constructing' a representation such that it captures framing effects in choice behavior. A representation is a mental image of reality. A (classical) consistency requirement is that representations be 'compatible', i.e., it must be possible to derive the different representations from a single underlying reality.[19] In terms of the theory developed in this paper this is equivalent to requiring that all FR operators are commuting. In contrast, the HSM allows for 'incompatible' representations i.e., that cannot be consolidated into one single consistent picture of the decision situation although they *do* represent the *same* reality.[20] This is because we do not assume that the picture pre-exists the 'act of representation.' The quantum model of reality allows us to understand framing effects without assuming that the individual is 'boundedly rational', which is consistent with highly rational agents exhibiting framing effects. In the HSM framework, framing effects arise 'naturally' as a consequence of (initial) indeterminacy of the agent's representation of the decision situation. So far this is only a descriptive analysis. In order for it to gain some predictive power we must identify those properties of frame operators that are common to a class of DSs. One example of such a property is that the choice situation is mentally represented as a test of personal

[19]This is reminiscient of the consistency requirement when modelling subjective beliefs.

[20]The representations are 'revealed' incompatible in the sense that agents make different choices in game-theoretically equivalent situations. In our framework incompatibility of representations is expressed as the non-commutativity of the corresponding framing operator (representations are eigenvalues belonging to a framing operator).

qualities (cf. $\widetilde{B}$ above). Another approach is to focus on the mental representation stage. We could experimentally investigate the effect of a sequence of frame operators in different orders on choice.

3.2.1 Testing the theory

In this section we propose two experimental tests of the proposed theory.

Experiment 1: Superposition of states of mind Consider two identical populations of agents I and II. We propose an experiment where the two populations are invited to play a Prisoners' Dilemma game against a hidden player. Before playing the PD game,the agents from the first population are invited to answer a question by YES or NO.[21] For the sake of the argument it does not matter what the question is, provided it bears on a relevant type characteristic. It could, for instance, be a question interpreted as revealing whether the agent is 'selfish' or 'altruistic'. The agents from the second population play the PD game without any 'preparation'.

Suppose that we observe a proportion α of YES-answers in the first population of agents (and a proportion $(1-\alpha)$ of NO-answers). Denote by B the operator representing the question. Now, the agents are invited to play the PD game (operator A).

Assume that we find that a proportion β of agents of the 'YES-type' (agents who have answered YES) choose to cooperate ($(1-\beta)$ choose to defect) and a proportion γ of agents of the 'NO-type' choose to cooperate ($(1-\gamma)$ to defect). The proportion of agents of the first population choosing C is thus

$$p_I(C) = \alpha\beta + (1-\alpha)\gamma.$$

The second part of the experiment is performed on the other population of agents who play the PD game directly. From a classical point of view, agents are either YES-type or NO-type independently of the fact that they have answered the questionnaire (cf. stable innate preferences). Since the two populations are identical, we should be able to apply the conditional probability formula to obtain

$$p_{II}(C) = p(a)p(C|a) + p(s)p(C|s) = \alpha\beta + (1-\alpha)\gamma.$$

[21] Such YES/NO questions are the basis of Mackey's Quantum Logic. Any observable can be expressed as the meet of reciprocally orthogonal YES/NO questions.

Assume that we find that the proportion of agents in population II who choose C is $p'(C)$, which is significantly different from $p_{II}(C)$ above. The only way to explain these results within a classical framework would be to give up the conditional probability formula and hence to assume that the agents, though remaining in a definite state (YES-type or NO-type) nevertheless change the way they play the PD game solely by the virtue of having answered the question. This assumption does not seem to be consistent with the idea of an agent having stable innate preferences.

We propose that at the beginning, the agent is in a superposition of two types: YES-type and NO-type. Both modes of behavior are 'latent' and have the potential to be observed. As the agent answers the question, this 'hesitation' is resolved: one of the two behavioral inclinations (propensities to act) emerges as his type. Before this operation, the two modes of behavior affect his choice (in the PD game) but as soon as one mode has manifested itself, only this mode is active. If in fact the results with respect to the choice in the PD game turn out to be different in the two situations (having answered the question or not), the proposed experiment provides support for the hypothesis that preferences are indeterminate, i.e., agents can be represented by a superposition of types.

Experiment 2: Order matters We consider three different DSs: the first is the familiar Prisoners' Dilemma (PD); the second is the Dictator Game (DG) with the two options 'sharing equally' and 'sharing selfishly'.[22] The third DS that we consider is the responder's decision situation in the Ultimatum Game (UG).[23] Suppose that we have empirically established that order matters for the two pairs of DSs (PD, DG) and (DG,UG). According to our theory, these DSs must be represented by pairs of non-commuting operators. Then, unless the behavior types in the PD and UG are perfectly correlated, (i.e., the players who choose 'cooperate' in the PD either always select the action 'accept' in UG or they always select 'refuse'), our theory

[22] The Dictator Game is a two-players game in which one player, the Dictator, decides on a split of an amount of money, say 100 units, between himself and the other player who is actually a dummy.

[23] The Ultimatum Game is played as follows. The first mover (here a hidden player) proposes a given split of a known sum of money. The decision of the responder is whether to accept or to reject the split. In the case of rejection, the payoff is zero to both players, in particular to our decision-maker – the responder.

predicts that order matters in the play of the pair of DSs (PD,UG).[24]

In order to test this prediction we propose the following experiment. We first establish whether or not order matters between the first two pairs of DSs (PD,DG) and (DG,UG) using the following procedure: we take two populations of *identical* individuals (assumed to be represented by the same state vector). The first population is invited to play the following sequence: first the PD (against a hidden player as in experiment 1) and thereafter the DG (against a hidden third player). The individuals from the second population are invited to play the same DSs but in the reverse order: first the DG and thereafter the PD. If the distributions of choices turn out to be significantly different we have established that order matters when playing the PD and DG. We perform a similar test for (DG,UG) (with another set of participants). If we do observe that order matters even in this case, we move on to the next step.

The next step of the experiment is to test whether the types in PD and UG are perfectly correlated, and we do this by selecting a population of agents who have chosen 'cooperate' in the PD. Assuming that there is no evidence of perfect correlation, we can then move to the last phase of the experiment, namely, testing the prediction that PD and UG are non-commuting decision situations. We do that by the same procedure we used to test the non-commutativity of (PD,DG) and (DG,UG) (of course with another set of participants). If we do in fact observe that the distributions of choices are significantly different it will be consistent with our theory indeterminacy of preferences.

Our objective is to take two examples of experiments that involve a common decision situation and for which two different behavioral theories have been proposed. Our contribution would be to link the two experimental phenomena with a 'cross-prediction' along the lines described above. If the prediction turns out to be correct but cannot be explained by either one of those theories, we will have demonstrated that our theory has the potential to both unify some of the existing body of behavioral economics and generate new results.

[24]Technically speaking, given three operators A, B, C (with two eigenvectors each), if both (A, B) and (B, C) are pairs of non-commuting operators, then the eigenvectors of A as well as those of C form a basis of the same two-dimensional space. If, in addition, A and C commute, then A and C must have the same eigenvectors. The two operators differ only in terms of the names given to the eigenvalues. In other words the types in A are perfectly correlated with the types in C.

4 Concluding remarks

In this paper we propose an approach to decision-making that has the potential to provide a unified framework for explaining a variety of so-called 'behavioral anomalies'. The basic idea is that preferences, besides being typically unknown (to other agents), are also indeterminate. A main implication of the indeterminacy of preferences is that a choice situation does not merely reveal the (fixed) preferences of an agent, but also contributes to constructing them. Consequently, in order to be able to make predictions one also must account for the context in which preferences are expressed. The Hilbert space model we propose provides a mathematical model for making such contextual predictions.

We argue that this abstract model may be an appropriate extension of Harsanyi's model of uncertainty, which accommodates empirically observed phenomena, like cognitive dissonance, context dependency of preferences or framing effects.

This preliminary investigation is a first step of a larger research program intended to investigate the implications of the indeterminacy hypothesis in game theory and economics. Those implications will then be tested experimentally.

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5 Appendix: Elements of Quantum Mechanics

5.1 States and Observables

In Quantum Mechanics the state of a system is represented by a vector $|\psi\rangle$ in a Hilbert space $\mathcal{H}$. According to the superposition principle, every complex linear combination of state vectors is a state vector. A Hermitian operator called an observable is associated to each physical property of the system.

Theorem 1

A Hermitian operator A has the following properties:

- Its eigenvalues are real.

- Two eigenvectors corresponding to different eigenvalues are orthogonal.

- There is an orthonormal basis of the relevant Hilbert space formed with the eigenvectors of A.

Let us call $|v_1\rangle, |v_2\rangle, ..., |v_n\rangle$ the normalized eigenvectors of A forming a basis of $\mathcal{H}$. They are associated with eigenvalues $\alpha_{1,}\alpha_2, ..., \alpha_n$, so: $A|v_i\rangle = \alpha_i |v_i\rangle$. The eigenvalues can possibly be degenerated, i.e., for some i and j, $\alpha_i = \alpha_{j.}$ This means that there is more than one linearly independent eigenvector associated with the same eigenvalue. The number of these eigenvectors defines the degree of degeneracy of the eigenvalue which in turn defines the dimension of the eigensubspace spanned by these eigenvectors. In this case, the orthonormal basis of $\mathcal{H}$ is not unique because it is possible to replace the eigenvectors associated to the same eigenvalue by any complex linear combination of them to get another orthonormal basis. When an observable A has no degenerated eigenvalue there is a unique orthonormal basis of $\mathcal{H}$ formed with its eigenvectors. In this case (see below), it is by itself a Complete Set of Commuting Observables.

Theorem 2

If two observables A and B commute there is an orthonormal basis of $\mathcal{H}$ formed by eigenvectors common to A and B.

Let A be an observable with at least one degenerated eigenvalue and B another observable commuting with A. There is no unique orthonormal basis formed by A eigenvectors. But there is an orthonormal basis of the relevant

Hilbert space formed by eigenvectors common to A and B. By definition, $\{A, B\}$ is a Complete Set of Commuting Observables (CSCO) if this basis is unique. Generally, a set of observables $\{A, B, ...\}$ is said to be a CSCO if there is a unique orthonormal basis formed by eigenvectors common to all the observables of the set.

5.2 The measurement

An observable A is associated to each physical property of a system S. Let $|v_1\rangle, |v_2\rangle, ..., |v_n\rangle$ be the normalized eigenvectors of A associated respectively with eigenvalues $\alpha_{1,}\alpha_2, ..., \alpha_n$ and forming a basis of the relevant state space. Assume the system is in the normalized state $|\psi\rangle$. A measurement of A on S obeys the following rules, collectively called 'Wave Packet Reduction Principle' (the Reduction Principle).

Reduction Principle

1. When a measurement of the physical property associated with an observable A is made on a system S in a state $|\psi\rangle$, the result only can be one of the eigenvalues of A.

2. The probability to get the non-degenerated value α_i is $P(\alpha_i) = |\langle v_i|\psi\rangle|^2$.

3. If the eigenvalue is degenerated then the probability is the sum over the eigenvectors associated with this eigenvalue: $P(\alpha_i) = \sum \left|\left\langle \nu_i^j|\psi\right\rangle\right|^2$.

4. If the measurement of A on a system S in the state $|\psi\rangle$ has given the result α_i then the state of the system immediately after the measurement is the normalized projection of $|\psi\rangle$ onto the eigensubspace of the relevant Hilbert space associated with α_i. If the eigenvalue is not degenerated then the state of the system after the measurement is the normalized eigenvector associated with the eigenvalue.

If two observables A and B commute then it is possible to measure both simultaneously: the measurement of A is not altered by the measurement of B. This means that measuring B after measuring A does not change the value obtained for A. If we again measure A after a measurement of B, we again get the same value for A. Both observables can have a definite value.

5.2.1 Interferences

The archetypal example of interferences in quantum mechanics is given by the famous two-slits experiment.[25] A parallel beam of photons falls on a diaphragm with two parallel slits and strikes a photographic plate. A typical interference pattern showing alternate bright and dark rays can be seen. If one slit is shut then the previous figure becomes a bright line in front of the open slit. This is perfectly understandable if we consider photons as waves, as it the assumption is in classical electromagnetism. The explanation is based on the fact that when both slits are open, one part of the beam goes through one slit and the other part through the other slit. Then, when the two beams join on the plate, they interfere constructively (giving bright rays) or destructively (giving dark ones), depending on the difference in length of the paths they have followed. But a difficulty arises if photons are considered as particles, as can be the case in quantum mechanics. Indeed, it is possible to decrease the intensity of the beam so as to have only one photon travelling at a time. In this case, if we observe the slits in order to detect when a photon passes through (for example, by installing a photodetector in front of the slits), it is possible to see that each photon goes through only one slit. It is never the case that a photon splits to go through both slits. The photons behave like particles. Actually, the same experiment was done with electrons instead of photons, with the same result. If we do the experiment this way with electrons (observing which slit the electrons go through, i.e., sending light through each slit to "see" the electrons), we see that each electron goes through just one slit and, in this case, we get no interference. If we repeat the same experiment without observing which slit the electrons pass through then we recover the interference pattern. Thus, the simple fact that we observe which slit the electron goes through destroys the interference pattern (two single slit patterns are observed). The quantum explanation is based on the assumption that when we don't observe through which slit the electron has gone then its state is a superposition of both states "gone through slit 1" and "gone through slit 2"[26], while when we observe it, its state collapses onto one of these states. In the first case, the position measurement is made on electrons in the superposed state and gives an interference pattern since both states are manifested in the measurement. In the second case, the position is

[25] See, e.g., Feynman (1965) for a very clear presentation.

[26] This doesn't mean that the photon actually went through both slits. This state simply can't be interpretated from a classical point of view.

measured on electrons in a definite state and no interference arises. In other words, when only slit 1 is open we get a spectrum, say S_1 (and S_2 when only slit 2 is open). We expect to get a spectrum S_{12} that sums of the two previous spectra when both slits are open, but this is not the case: $S_{12} \neq S_1 + S_2$.

5.3 HSM and the classic probabilistic model

This Appendix is intended to clarify the relationship between the proposed Hilbert space representation and the classical, probabilistic representation.

Our typical setup is that of two observables, A with eigenvalues $\{1_A, 2_A, ..., n_A\}$, and B with eigenvalues $\{1_B, 2_B, ..., m_B\}$. So to treat both observables we express the Hilbert space $\mathcal{H}$ as a direct sum in two different ways:

$$\mathcal{H} = \mathcal{H}_{1_A} \oplus, ..., \oplus \mathcal{H}_{n_A} \text{ and } \mathcal{H} = \mathcal{H}_{1_B} \oplus, ..., \oplus \mathcal{H}_{m_B},$$

where $\mathcal{H}_{i_A}$ is the subspace of eigenvectors of A with eigenvalue i_A (similarly with $\mathcal{H}_{j_B}$). Given the state $|\psi\rangle \in \mathcal{H}$ of the agent under consideration, it can be represented as

$$|\psi\rangle = \sum_i \lambda_i |i_{A\psi}\rangle, \quad \sum_i \lambda_i^2 = 1, \quad |i_{A\psi}\rangle \in \mathcal{H}_{i_A}, \ i = 1, ..., n. \tag{7}$$

Similarly,

$$|\psi\rangle = \sum_j \nu_j |j_{B\psi}\rangle, \quad \sum_i \nu_j^2 = 1 \quad |j_{B\psi}\rangle \in \mathcal{H}_{j_B}, \ j = 1, ..., m. \tag{8}$$

Now, states $|j_{B\psi}\rangle \in \mathcal{H}_{j_B}$ of eq. (8) can also be written as a superposition of the form (7):

$$|j_{B\psi}\rangle = \sum_i \mu_{ij} |i_{Aj_B}\rangle, \quad \sum_i \mu_{ij}^2 = 1 \quad |i_{Aj_B}\rangle \in \mathcal{H}_{i_A}, \ i = 1, ..., n. \tag{9}$$

Note that since the vectors $|i_{A\psi}\rangle$ and $|i_{Aj_B}\rangle$, are both in $\mathcal{H}_{i_A}$, they are eigenvectors of A with the same eigenvalue i_A. The *non-commutativity* of A and B is the violation of the equation

$$p_A(i_A) = p_{AB}(i_A), \quad i = 1, ...n, \tag{10}$$

which is $\lambda_i^2 = \sum_j \mu_{ij}^2 \nu_j^2$. That is, the probability distribution on the outcomes of the measurement A may be affected by the fact that we measured B before A.

Can this be modelled in standard probability theory? The answer is obviously yes. To do that let A be a random variable with values in $\{1_A, 2_A, ..., n_A,\}$ and with probabilities $p_A(i_A) = \alpha_i \equiv \lambda_i^2$, $i = 1, ..., n$, which will be the equivalent of eq. (7). Similarly, the equivalent of eq. (8) is obtained if B is a random variable with values in $\{1_B, 2_B, ..., m_B\}$ and with probabilities $p_B(j_B) = \beta_j \equiv \nu_j^2$, $j = 1, ..., m$.

What is the analogue of eq. (9)? For each value j_B of B there is a probability distribution p_{Aj_B} of the random variable A given by

$$p_{Aj_B}(i_A) = \gamma_{ij} \equiv \mu_{ij}^2; \quad i = 1, ...n, \qquad j = 1, ..., m. \tag{11}$$

Then, when measuring B and then A, the distribution of outcomes is given by

$$p_{AB}(i_A) = \sum_j \beta_j \gamma_{ij} \quad i = 1, ..., n.$$

Let $\alpha = (\alpha_1, ..., \alpha_n)$, $\beta = (\beta_1, ..., \beta_m)$, and $\gamma = \left(\gamma_{ij}\right)_{i=1,...,n,\, j=1,...,m.}$, then the analogue of eq. (10) is

$$\alpha = \beta\gamma, \tag{12}$$

which is generally not satisfied unless we impose restrictions on α, β, and γ.[27]

The following facts are easily verified:

- The equality (12) holds if and only if there is a probability distribution p on $\{(i_A, j_B);\ i = 1, ..., n,\ j = 1, ..., m\}$ whose marginal distribution of A and B are p_A and p_B respectively.

  In such a case p_{Aj_B} is just the conditional probability

$$p_{Aj_B}(i_A) = p(i_A | j_B) \tag{13}$$

and eq. (12) is then just the fundamental equation of conditional probability, namely, for $i = 1, ..., n$,

$$p(i_A) = \sum_j p(j_B) p(i_A | j_B)$$

- In the HSM framework eq. (10) holds if and only if the observables A and B commute, and we conclude that

[27] For example, take $\alpha = \beta = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\gamma = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \end{pmatrix}$, then $\gamma\beta = \left(\frac{3}{8}, \frac{5}{8}\right)$.

1. Two commuting decision situations A and B can be modelled by *one* standard probability space whose elementary events are the pairs (i_A, j_B) of values of the two observables.

2. Two non-commuting DSs A and B cannot be modelled by a single probability space. In particular, $p_{Aj_B}(i_A)$ *is not a conditional probability* in the formal sense since there is no underlying probability space in which the probabilities $p(i_A, j_B)$ are defined.